Theor Chem Acc (2003) 110: 403–409
DOI 10.1007/s00214-003-0494-7

Theoretical
Chemistry Accounts

Regular article

New isospectral generalized potentials

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Received: 8 September 2002 / Accepted: 15 May 2003 / Published online: 1 December 2003

Abstract. In quantum chemistry, supersymmetry, shape invariance and intertwining techniques are used to determine the class of potentials that are solvable as well as to find their isospectral and generalized partners. To do that, it is necessary to have the corresponding Witten superpotential defined by $W(x) = \psi'/\psi$ where ψ is a particular wavefunction of the Hamiltonian under study. In this work, we propose an alternative way to express the Witten superpotential in terms of reciprocal wavefunctions. Thus, when this new definition of $W(x)$ is used as an ansatz in the Riccati equation involved, one is led to a potential identical to that resulting from the use of the standard Darboux transform, which means that it is possibly the generalization of it. Moreover, the generalization of the new Witten superpotential gives rise to a new generalized isospectral potential other than that obtained from the generalized Darboux transform. As an example of an application of the proposed approach, we found the new generalized isospectral potentials that correspond to the one-dimensional free particle, harmonic oscillator and Morse potential models. Also, owing to the fact that the proposed method is general our proposal can be used straightforwardly to obtain new, exactly solvable potentials as well as to find their isospectral generalized partners which can be used advantageously in the modeling of important quantum chemical applications.

Keywords: Supersymmetry – Isospectral Hamiltonians – Darboux Transforms

1 Introduction

The physical and chemical microscopic properties of a system, at the quantum level, are well characterized through the Schrödinger equation [1]. Consequently, the usefulness of such a second-order differential equation depends on the degree of solvability that it has in relation to the potential involved [2]. That is, the study of exactly solvable quantum mechanical problems is a very interesting subject which has attracted increasing interest in theoretical sciences [3, 4]. Actually, to find solvable quantum problems the explicit knowledge of the potential under study becomes necessary in order to determine the most appropriate procedure, analytical or algebraic, of finding the corresponding eigenvalues [5]. The same occurs with standard isospectral potentials where the Riccati equation involved can be solved for a particular potential having the property of shape invariance [6]. Also, another way of obtaining new exactly solvable potentials consists of looking for pairs of quantum Hamiltonians coupled supersymmetrically by an intertwining operator [7, 14] or, as proposed recently by Morales et al. [8], to use an ansatz as a particular solution of the corresponding Riccati equation with the purpose to identify the potential under study. In consequence, owing to the fact that this start corresponds to the Witten [9] superpotential $W(x) = \psi'/\psi$, where ψ is a particular wavefunction of the Schrödinger equation involved, the wavefunctions and $W(x)$ can be generalized giving way to new isospectral potentials which are partners of the former potential. With all these elements taken into consideration, in the present work we show that the Witten superpotential can also be given in terms of $\varphi = \frac{1}{\psi}$, which permits us to obtain new generalized Witten superpotentials as well as new generalized wavefunctions which are involved with new generalized isospectral potentials. For that, in the next section we consider the generalization of those isospectral potentials associated with solvable potentials identified by means of the Witten superpotential, in terms of ψ and φ, as a particular solution of the Riccati equation concerned. Next, in order to show the usefulness of the proposed approach in the search for new generalized isospectral potentials, we consider explicitly the application of our proposal to the case of the one-dimensional free particle, harmonic

From the Proceedings of the 28th Congreso de Químicos Teóricos de Expresíon Latina (QUITEL 2002)

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oscillator and Morse potential models. However, owing to the fact that the proposed method is general it can be used directly to obtain new exactly solvable potentials as well as to find their isospectral and generalized partners which can be useful in the treatment of different quantum chemical applications.

2 Exactly solvable partner isospectral potentials

The time-independent Schrödinger equation for one-dimensional potentials, $V(x)$, is given by

$$-\psi_n''(x) + V(x)\psi_n(x) = \lambda_n\psi_n(x) \ . \tag{1}$$

By assuming that $\psi_0(x)$ is a particular solution of Eq. (1) it follows that the Witten superpotential [9]

$$W_0(x) = \psi_0'(x)/\psi_0(x) \ , \tag{2}$$

satisfies

$$W_0^2(x) + W_0'(x) = \frac{\psi_0''(x)}{\psi_0(x)} \ , \tag{3}$$

where

$$\psi_0(x) = \exp\left(\int^x W_0(x)\mathrm{d}x\right) \ . \tag{4}$$

The Schrödinger equation for $\psi_0(x)$ is then

$$-\psi_0''(x) + V(x)\psi_0(x) = \lambda_0\psi_0(x) \ , \tag{5}$$

where λ_0 is the so-called factorization energy and from which we obtain trivially

$$\frac{\psi_0''(x)}{\psi_0(x)} = V(x) - \lambda_0 \ . \tag{6}$$

That is, by using Eq. (6) in Eq. (3) it is found that the Witten superpotential satisfies the Riccati equation

$$V(x) = V^+(x) = W_0^2(x) + W_0'(x) + \lambda_0 \ , \tag{7}$$

where we have renamed $V(x)$ in order to differentiate it from the partner supersymmetric potential $V^-(x)$ that we obtain next. For that purpose, we note that the Witten superpotential given in Eq. (2) can be rewritten as

$$W_0(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\ln[\varphi_0(x)] \ , \tag{8}$$

where $\varphi_0 = \frac{1}{\psi_0}$, indicating that

$$\varphi_0(x) = \exp\left(-\int^x W_0(x)\mathrm{d}x\right) \ . \tag{9}$$

Thus, following a similar procedure to the one given previously for the potential $V^+(x)$, in this case the Schrödinger equation for φ_0 comes from

$$W_0^2(x) - W_0'(x) = \frac{\varphi_0''(x)}{\varphi_0(x)} \ , \tag{10}$$

on condition that we have the partner supersymmetric potential

$$V^-(x) = W_0^2(x) - W_0'(x) + \lambda_0 \ , \tag{11}$$

which means that

$$\frac{\varphi_0''(x)}{\varphi_0(x)} = V^-(x) - \lambda_0 \ . \tag{12}$$

That is,

$$V^-(x) = V^+(x) - 2W_0'(x) \ ; \tag{13}$$

therefore

$$W_0'(x) = \frac{1}{2}[V^+(x) - V^-(x)] \ , \tag{14}$$

and

$$W_0^2(x) = \frac{1}{2}[V^+(x) + V^-(x)] - \lambda_0 \ . \tag{15}$$

At this point, two facts should be noted: first, Eq. (13) is the equivalent to the Gendenshtein [10] condition of shape invariance for solvable one-dimensional potential models; and second, the potential $V^-(x)$ matches with the so-called Darboux potential which comes from the standard Darboux transform [11].

Concerning the solutions of the Riccati equations involved, it becomes clear that Eqs. (7) and (11) have the same particular solution, $W_0(x)$, although a different general solution exists for each potential. That is, the general solution for the potential $V^+(x)$ of the Riccati Eq. (7) is given by

$$W_\mathrm{g}^+(x) = W_0(x) + \frac{b}{\rho(x)} \ , \tag{16}$$

where

$$\rho(x) = \mathrm{e}^{2\int^x W_0(x)\mathrm{d}x}\left(\gamma + b\int^x \mathrm{e}^{-2\int^x W_0(x)\mathrm{d}x}\mathrm{d}x\right) \ , \tag{17}$$

with γ and b integration constants and where we have used the lower index g to indicate generalized. Similarly, the general solution for the partner supersymmetric potential $V^-(x)$ of the Riccati Eq. (11) is given by

$$W_\mathrm{g}^-(x) = W_0(x) + \frac{a}{\eta(x)} \ , \tag{18}$$

where

$$\eta(x) = \mathrm{e}^{-2\int^x W_0(x)\mathrm{d}x}\left(\beta - a\int^x \mathrm{e}^{2\int^x W_0(x)\mathrm{d}x}\mathrm{d}x\right) \ , \tag{19}$$

with β and a integration constants. Thus, owing to the fact that Eq. (16) can be rewritten as

$$W_\mathrm{g}^+(x) = W_0(x) + \frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\gamma + b\int^x \mathrm{e}^{-2\int^x W_0(x)\mathrm{d}x}\mathrm{d}x\right) \ , \tag{20}$$

the substitution of $W_0(x)$, given in Eq. (2), leads to

$$\begin{aligned} W_\mathrm{g}^+(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\ln\psi_0(x) \\ &\quad + \frac{\mathrm{d}}{\mathrm{d}x}\ln\left[\gamma + b\int^x\left(\mathrm{e}^{-\int^x W_0(x)\mathrm{d}x}\right)^2\mathrm{d}x\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\ln\psi_{\mathrm{g}_0}(x) \ , \end{aligned} \tag{21}$$

where

$$\psi_{g_0}(x) = \psi_0(x)\left(\gamma + b\int^x \frac{dx}{\psi_0^2(x)}\right) \quad . \tag{22}$$

It is important to stress that this last relationship is a second solution of the Schrödinger equation for the former potential $V^+(x)$, in good agreement with Korolev [12]. Similarly, $W_g^-(x)$ is now

$$W_g^-(x) = W_0(x) - \frac{d}{dx}\ln\left(\beta - a\int^x e^{2\int^x W_0(x)dx}dx\right) \tag{23}$$

or explicitly, after using Eq. (8),

$$\begin{aligned} W_g^-(x) &= -\frac{d}{dx}\ln\varphi_0(x) - \frac{d}{dx}\ln\left(\beta - a\int^x \frac{dx}{\varphi_0^2}\right) \\ &= -\frac{d}{dx}\ln\varphi_{g_0}(x) \quad , \end{aligned} \tag{24}$$

where

$$\varphi_{g_0}(x) = \varphi_0(x)\left(\beta - a\int^x \frac{dx}{\varphi_0^2(x)}\right) \quad . \tag{25}$$

As before, in this case $\varphi_{g_0}(x)$ is the second solution of the Schrödinger equation where the generalized potential

$$V_g^-(x) = V^-(x) = V^+(x) - 2W_0'(x) \tag{26}$$

is an isospectral partner of the former potential $V^+(x)$.

Finally, it is important to underline the existence of the reciprocal wavefunctions

$$\chi_{g_0}(x) = \frac{1}{\psi_{g_0}(x)} \tag{27}$$

and

$$h_{g_0}(x) = \frac{1}{\varphi_{g_0}(x)} \tag{28}$$

with properties

$$W_g^+(x) = -\frac{\chi_{g_0}'(x)}{\chi_{g_0}(x)}; \quad W_g^+(x)^2 - \frac{dW_g^+(x)}{dx} = \frac{\chi_{g_0}''(x)}{\chi_{g_0}(x)} \tag{29}$$

and

$$W_g^-(x) = \frac{h_{g_0}'(x)}{h_{g_0}(x)}; \quad W_g^-(x)^2 + \frac{dW_g^-(x)}{dx} = \frac{h_{g_0}''(x)}{h_{g_0}(x)} \quad . \tag{30}$$

Similarly to the previous cases, the $\chi_{g_0}(x)$ and $h_{g_0}(x)$ functions are solutions of the Schrödinger equation involved with the isospectral potentials

$$\begin{aligned} \mathscr{V}^+(x) &= W_g^+(x)^2 - \frac{dW_g^+(x)}{dx} + \lambda_0 \\ &= V^+(x) - 2\frac{dW_g^+(x)}{dx} \quad , \end{aligned} \tag{31}$$

and

$$\begin{aligned} \mathscr{V}^-(x) &= W_g^-(x)^2 + \frac{dW_g^-(x)}{dx} + \lambda_0 \\ &= V^-(x) + 2\frac{dW_g^-(x)}{dx} \quad . \end{aligned} \tag{32}$$

It should be observed that $\mathscr{V}^+(x)$ is identical to the generalized Darboux potential [13]:

$$\mathscr{V}^+(x) = V^-(x) - 2\frac{d}{dx}\left(\frac{b}{\rho(x)}\right) \quad , \tag{33}$$

fulfilling the Schrödinger equation

$$-\chi_{g_0}''(x) + \mathscr{V}^+(x)\chi_{g_0}(x) = \lambda_0\chi_{g_0}(x) \quad , \tag{34}$$

where

$$\chi_{g_0}(x) = \frac{\varphi_0(x)}{\gamma + b\int^x \varphi_0^2(x)dx} \quad . \tag{35}$$

Similarly, the new potential $\mathscr{V}^-(x)$ satisfies the Schrödinger equation

$$-h_{g_0}''(x) + \mathscr{V}^-(x)h_{g_0}(x) = \lambda_0 h_{g_0}(x) \tag{36}$$

according to

$$h_{g_0}(x) = \frac{\psi_0(x)}{\beta - a\int^x \psi_0^2(x)dx} \quad . \tag{37}$$

In short, we have four different isospectral potentials: the former, the Darboux potential, the generalized Darboux potential and another new, generalized potential. Also, it is worth mentioning that Roy and Roychoudhury [14] constructed a sequence of supersymmetric quantum mechanical Hamiltonians leading to the isospectral potentials that come from our Eq. (3). However, they did not obtain the new isospectral potentials derived from our Eq. (10) owing to the fact that they did not solve for the general solution of the corresponding Riccati equation. From a general point of view, in Table 1 we have shown our proposed algorithm to find the aforementioned four different isospectral potentials. On the other hand, from a particular situation, in the next section we are going to apply the proposed method for obtaining those new isospectral potentials associated with some standard one-dimensional potential models.

3 New generalized isospectral potentials

As a useful application of the proposed approach we will obtain next new generalized isospectral potentials that are the partners of the standard one-dimensional free particle, harmonic oscillator and Morse potential models.

3.1 One-dimensional free particle potential

The one-dimensional free particle potential is probably the simplest potential that can be found in quantum

Table 1. Algorithm to obtain four different isospectral potentials

Witten superpotentials	Wavefunctions	Potential partners–Riccati equations
$W_o(x) = \frac{\Psi'_o(x)}{\Psi_o(x)}$	Ψ_o	$V^+(x) = W'_o(x) + W_o^2(x) + \lambda_o$
$W_g^+(x) = \frac{\Psi'_{g_o}(x)}{\Psi_{g_o}(x)}$	$\Psi_{g_o} = \Psi_o\left(\gamma + b\int\frac{dx}{\Psi_o^2}\right)$	$V_g^+(x) = W_g^{+'}(x) + W_g^{+^2}(x) + \lambda_o = V^+(x)$
$W_g^+(x) = -\frac{\chi'(x)}{\chi(x)}$	$\chi = \frac{1}{\Psi_{g_o}}$	$\mathscr{V}_g^+(x) = -W_g^{+'}(x) + W^{+^2}(x) + \lambda_o$
$W_o(x) = -\frac{\varphi'_o(x)}{\varphi_o(x)}$	$\varphi_o = \frac{1}{\Psi_o}$	$V^-(x) = -W'_o(x) + W_o^2(x) + \lambda_o$
$W_g^-(x) = -\frac{\varphi'_{g_o}(x)}{\varphi_{g_o}(x)}$	$\varphi_{g_o} = \varphi_o\left(\beta - a\int\frac{dx}{\varphi_o^2}\right)$	$V_g^-(x) = V^+(x) - 2W'_o(x) = V^-(x)$
$W_g^- = \frac{h'(x)}{h(x)}$	$h = \frac{1}{\varphi_{g_o}}$	$\mathscr{V}_g^-(x) = W_g^{-'}(x) + W_g^{-^2}(x) + \lambda_o$

studies. In this work, this potential comes from the simplest choice of the Witten superpotential $W_0(x) = i\alpha$, where $\alpha \in \mathbb{R}$, as a particular solution of the corresponding Riccati relationship. In fact, this particular solution of Eqs. (7) and (11) indicates that $V^+ = V^- = 0$ on condition that we have $\lambda_0 = \alpha^2$. Obviously, the corresponding wavefunction of the Schrödinger equation is in this case, according to Eq. (4), given by

$$\psi_0 = \exp(i\alpha x) \ . \tag{38}$$

In order to generalize this wavefunction as well as the corresponding Witten superpotential, we use Eqs. (22) and (21) respectively. Thus, after some cumbersome algebra we obtain

$$\psi_{g_0}(x) = \frac{b\sqrt{g}}{\alpha}\sin(\alpha x - i\ln\sqrt{g}) \ , \tag{39}$$

where $g = 2i\alpha\gamma/b$, and

$$W_g^+(x) = \alpha\cot(\alpha x - i\ln\sqrt{g}) \ . \tag{40}$$

Consequently, the associated generalized isospectral potential becomes

$$\mathscr{V}^+(x) = 2\alpha^2\csc^2(\alpha x - i\ln\sqrt{g}) \tag{41}$$

with the wavefunction given by

$$\chi_{g_0}(x) = \frac{\alpha}{b\sqrt{g}}\csc(\alpha x - i\ln\sqrt{g}) \ . \tag{42}$$

As can be seen, potential $\mathscr{V}^+(x)$ given in Eq. (41) contains some particular cases depending on the choice of γ and b. For example, in the case of $\gamma = 1/2$ and $b = -i\alpha$, Eq. (39) can be rewritten as

$$\psi_{g_0}(x) = \cos(\alpha x) \ , \tag{43}$$

for which the Witten superpotential becomes

$$W_g^+(x) = -\alpha\tan(\alpha x) \ , \tag{44}$$

leading to the new generalized isospectral potential

$$\mathscr{V}^+(x) = 2\alpha^2\sec^2(\alpha x) \tag{45}$$

with wavefunction

$$\chi_{g_0}(x) = \sec(\alpha x) \ . \tag{46}$$

Similarly, when $\gamma = (2i)^{-1}$ and $b = \alpha$ are selected one has

$$\psi_{g_0}(x) = \sin(\alpha x) \ , \tag{47}$$

with Witten superpotential given by

$$W_g^+(x) = \alpha\cot(\alpha x) \ , \tag{48}$$

such that the corresponding generalized isospectral potential

$$\mathscr{V}^+(x) = 2\alpha^2\csc^2(\alpha x) \tag{49}$$

has the wavefunction

$$\chi_{g_0}(x) = \csc(\alpha x) \ . \tag{50}$$

On the other hand, for the reciprocal wavefunction

$$\varphi_0(x) = \exp(-i\alpha x) \ , \tag{51}$$

one has the corresponding generalized wavefunction

$$\varphi_{g_0}(x) = \frac{a\sqrt{r}}{i\alpha}\cos(\alpha x + i\ln\sqrt{r}) \ , \tag{52}$$

where $r = 2i\alpha\beta/a$, and the generalized Witten superpotential

$$W_g^-(x) = \alpha\tan(\alpha x + i\ln\sqrt{r}) \ . \tag{53}$$

Thus, the new generalized isospectral potential for the free particle potential model will be

$$\mathscr{V}^-(x) = 2\alpha^2\sec^2(\alpha x + i\ln\sqrt{r}) \ , \tag{54}$$

with the wavefunction given by

$$h_{g_0}(x) = \frac{i\alpha}{a\sqrt{r}}\sec(\alpha x + i\ln\sqrt{r}) \ . \tag{55}$$

Similarly to the previous case, the potential $\mathscr{V}^-(x)$ contains some particular cases depending on the choice of β and a. In fact, when $\beta = 1/2$ and $a = i\alpha$, Eq. (52) can be rewritten as

$$\varphi_{g_0}(x) = \cos(\alpha x) \ , \tag{56}$$

for which the corresponding Witten superpotential becomes

$$W_g^-(x) = \alpha\tan(\alpha x) \tag{57}$$

leading to the new generalized isospectral potential

$$\mathscr{V}^-(x) = 2\alpha^2\sec^2(\alpha x) \tag{58}$$

with wave function

$$h_{g_0}(x) = \sec(\alpha x) \ . \tag{59}$$

Finally, in the case $\beta = -(2i)^{-1}$ and $a = \alpha$, Eq. (52) becomes

$$\varphi_{g_0}(x) = \sin(\alpha x) \ , \tag{60}$$

with Witten superpotential given by

$$W_g^-(x) = -\alpha \cot(\alpha x) \ . \tag{61}$$

Thus, the corresponding generalized isospectral potential

$$\mathscr{V}^-(x) = 2\alpha^2 \csc^2(\alpha x) \tag{62}$$

has the wavefunction

$$h_{g_0}(x) = \csc(\alpha x) \ . \tag{63}$$

In any case, it will be clear that other new generalized free particle potentials can be obtained from other choices of γ and b, or β and a, for which, a potential like this, it will be considered as a family of isospectral potentials. For example, in Fig. 1 we have shown the free particle isospectral potentials $V^-(x), \mathscr{V}^-(x)$ and $\mathscr{V}^+(x)$, as well as the former $V^+(x)$. Notice that we have used $\alpha = 1$ and $V^-(x) \neq V^+(x) \neq 0$.

3.2 One-dimensional harmonic oscillator potential

The one-dimensional harmonic oscillator potential can be identified when we use the Witten superpotential $W_0(x) = -\alpha x$, where α is a constant, as particular solution of the Riccati Eq. (7). In fact, this ansatz leads to

$$V^+(x) = \alpha^2 x^2 \ , \tag{64}$$

on the condition that we have $\lambda_0 = \alpha$. This potential has wavefunctions

$$\psi_0(x) = \exp\left(-\frac{\alpha}{2}x^2\right) \tag{65}$$

and

$$\psi_{g_0}(x) = \exp\left(-\frac{\alpha}{2}x^2\right)\left(\gamma + b\int^x e^{\alpha x^2} \mathrm{d}x\right) \ . \tag{66}$$

Also, the use of Eq. (11) or Eq. (13) let us to find the Darboux harmonic oscillator potential

$$V^-(x) = V^+(x) + 2\alpha \ , \tag{67}$$

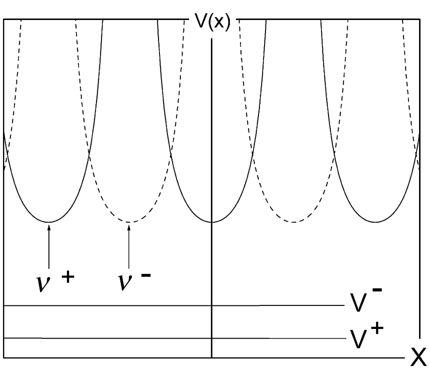

Fig. 1. Free particle isospectral potential partners. The potentials $\mathscr{V}^+(x)$ and $\mathscr{V}^-(x)$ are those obtained from Eqs. (45) and (62), respectively, using $\alpha = 1$

whose Schrödinger equation is satisfied by the wavefunctions

$$\varphi_0(x) = \exp\left(\frac{\alpha}{2}x^2\right) \tag{68}$$

and

$$\varphi_{g_0}(x) = \exp\left(\frac{\alpha}{2}x^2\right)\left(\beta - a\int^x e^{-\alpha x^2} \mathrm{d}x\right) \ . \tag{69}$$

In the same way, in order to obtain the corresponding generalized isospectral potentials it is necessary to use the generalized Witten superpotentials

$$W_g^+(x) = -\alpha x + \frac{b e^{\alpha x^2}}{\gamma + b\int^x e^{\alpha x^2} \mathrm{d}x} \tag{70}$$

and

$$W_g^-(x) = -\alpha x + \frac{a e^{-\alpha x^2}}{\beta - a\int^x e^{-\alpha x^2} \mathrm{d}x} \ . \tag{71}$$

That is, from Eqs. (31) and (32) the corresponding generalized harmonic oscillator potentials are

$$\mathscr{V}^+(x) = \alpha^2 x^2 + 2\alpha - 2\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{b e^{\alpha x^2}}{\gamma + b\int^x e^{\alpha x^2} \mathrm{d}x}\right) \ , \tag{72}$$

which is a new generalized isospectral harmonic oscillator potential, and

$$\mathscr{V}^-(x) = \alpha^2 x^2 + 2\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{a e^{-\alpha x^2}}{\beta - a\int^x e^{-\alpha x^2} \mathrm{d}x}\right) \ , \tag{73}$$

which has been identified by Mielnik [6] as a new isospectral potential with the harmonic oscillator spectrum. As already point out [6], the Mielnik's potential has no singularity if $|\beta| > \int_0^x a e^{-\alpha x^2} \mathrm{d}x$. Conversely, the new isospectral potential given in Eq. (72) has a singularity when $\gamma = -b\int_0^x e^{\alpha x^2} \mathrm{d}x$ as can be seen in Fig. 2 where, we have included all the other harmonic oscillator isospectral potentials. Finally, these potentials $\mathscr{V}^+(x)$ and $\mathscr{V}^-(x)$ have, respectively, the following wavefunctions

$$\chi_{g_0}(x) = \frac{e^{\frac{1}{2}\alpha x^2}}{\gamma + b\int^x e^{\alpha x^2} \mathrm{d}x} \tag{74}$$

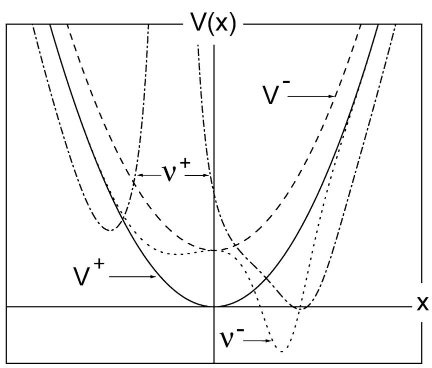

Fig. 2. Harmonic oscillator isospectral potential partners. $V^+(x)$ is from Eq. (64); $V^-(x)$ is from Eq. (67). For the potentials $\mathscr{V}^+(x)$ and $\mathscr{V}^-(x)$ we used Eqs. (72) and (73), respectively, with the parameters $\alpha = 1$, $\gamma = b$, $\beta = a$ and 0 as the lower integration limit

and

$$h_{g_0}(x) = \frac{e^{-\frac{1}{2}\alpha x^2}}{\beta - a\int^x e^{-\alpha x^2}dx} \quad . \tag{75}$$

3.3 Morse potential for s states

In order to identify the particular case of the standard Morse potential, we propose as a particular solution of the Riccati relationship, given in Eq. (7), the Witten superpotential

$$W_0(x) = Ae^{-\alpha x} - k \ , \tag{76}$$

where A, α and k are arbitrary constants to be determined. In fact, in this case one has

$$V^+(x) = A^2\left[e^{-2\alpha x} - \left(\frac{2k+\alpha}{A}\right)e^{-\alpha x}\right] + k^2 + \lambda_0 \ , \tag{77}$$

for which we can choose $A = k + \frac{\alpha}{2}$ and $\lambda_0 = -k^2$ in order to have

$$V^+(x) = \left(k + \frac{\alpha}{2}\right)^2\left(e^{-2\alpha x} - 2e^{-\alpha x}\right) \ . \tag{78}$$

This potential is to be compared with the Morse potential model

$$V_{\mathrm{Morse}}(x) = D\left(e^{-2\alpha x} - 2e^{-\alpha x}\right) \ , \tag{79}$$

in order to have $k = \sqrt{D} - \frac{\alpha}{2}$, where D is the depth of the well. The Schrödinger equation for this potential has wavefunctions

$$\psi_0(x) = e^{\left(\frac{\alpha}{2}-\sqrt{D}\right)x - \frac{\sqrt{D}}{\alpha}e^{-\alpha x}} \ , \tag{80}$$

as published elsewhere [15], and

$$\psi_{g_0}(x) = e^{\left(\frac{\alpha}{2}-\sqrt{D}\right)x - \frac{\sqrt{D}}{\alpha}e^{-\alpha x}} \times \left(\gamma + b\int \frac{dx}{e^{-2(\sqrt{D}-\alpha/2)x - \frac{2}{\alpha}\sqrt{D}e^{-\alpha x}}}\right) \ . \tag{81}$$

Furthermore, the corresponding Darboux potential is then given by

$$V^-(x) = D\left(e^{-2\alpha x} - 2e^{-\alpha x}\right) + 2\alpha\sqrt{D}e^{-\alpha x} \ , \tag{82}$$

which is in good agreement with Drigo Filho [16] and Morales et al. [13]. By the way, we want to point out some misprints in Ref. [13]: the number 2 occurring in the denominator of Eq. (70) should be deleted and k appearing in Eq. (74) should be changed by $2k$.

We note that potential $V^-(x)$ fulfills a Schrödinger equation with a wavefunction given by

$$\varphi_0(x) = e^{-\left(\frac{\alpha}{2}-\sqrt{D}\right)x + \frac{\sqrt{D}}{\alpha}e^{-\alpha x}} \tag{83}$$

and

$$\varphi_{g_0}(x) = e^{-\left(\frac{\alpha}{2}-\sqrt{D}\right)x + \frac{\sqrt{D}}{\alpha}e^{-\alpha x}} \times \left(\beta - a\int \frac{dx}{e^{-2(\frac{\alpha}{2}-\sqrt{D})x + \frac{2}{\alpha}\sqrt{D}e^{-\alpha x}}}\right) \ . \tag{84}$$

Similarly to the previous cases, in order to obtain the generalized Morse potentials $\mathcal{V}^+(x)$ and $\mathcal{V}^-(x)$, according to Eqs. (21) and (24) it is necessary to have the generalized Witten superpotentials $W_g^+(x)$ and $W_g^-(x)$, which are in this case given by

$$W_g^+(x) = \sqrt{D}e^{-\alpha x} - \sqrt{D} + \frac{\alpha}{2} + \frac{be^{2(\sqrt{D}-\alpha/2)x + \frac{2}{\alpha}\sqrt{D}e^{-\alpha x}}}{\gamma + b\int^x e^{2(\sqrt{D}-\alpha/2)x + \frac{2}{\alpha}\sqrt{D}e^{-\alpha x}}dx} \tag{85}$$

and

$$W_g^-(x) = \sqrt{D}e^{-\alpha x} - \sqrt{D} + \frac{\alpha}{2} + \frac{ae^{-2(\sqrt{D}-\alpha/2)x - \frac{2}{\alpha}\sqrt{D}e^{-\alpha x}}}{\beta - a\int^x e^{-2(\sqrt{D}-\alpha/2)x - \frac{2}{\alpha}\sqrt{D}e^{-\alpha x}}dx} \ . \tag{86}$$

Then, using Eqs. (31) and (32) we have the generalized Morse isospectral potentials

$$\mathcal{V}^+(x) = D(e^{-2\alpha x} - 2e^{-\alpha x}) + 2\alpha\sqrt{D}e^{-\alpha x} - 2\frac{d}{dx}\left(\frac{be^{2(\sqrt{D}-\alpha/2)x + \frac{2}{\alpha}\sqrt{D}e^{-\alpha x}}}{\gamma + b\int^x e^{2(\sqrt{D}-\alpha/2)x + \frac{2}{\alpha}\sqrt{D}e^{-\alpha x}}dx}\right) \tag{87}$$

and

$$\mathcal{V}^-(x) = D(e^{-2\alpha x} - 2e^{-\alpha x}) + 2\frac{d}{dx}\left(\frac{ae^{-2(\sqrt{D}-\alpha/2)x - \frac{2}{\alpha}\sqrt{D}e^{-\alpha x}}}{\beta - a\int^x e^{-2(\sqrt{D}-\alpha/2)x - \frac{2}{\alpha}\sqrt{D}e^{-\alpha x}}dx}\right) \ , \tag{88}$$

whose Hamiltonians follow from the wavefunctions

$$\chi_{g_0}(x) = \frac{e^{(\sqrt{D}-\alpha/2)x + \frac{1}{\alpha}\sqrt{D}e^{-\alpha x}}}{\gamma + b\int^x e^{2(\sqrt{D}-\alpha/2)x + \frac{2}{\alpha}\sqrt{D}e^{-\alpha x}}dx} \tag{89}$$

and

$$h_{g_0}(x) = \frac{e^{-(\sqrt{D}-\alpha/2)x - \frac{1}{\alpha}\sqrt{D}e^{-ax}}}{\beta - a\int^x e^{-2(\sqrt{D}-\alpha/2)x - \frac{2}{\alpha}\sqrt{D}e^{-ax}}dx} \ , \tag{90}$$

respectively. The former and the three Morse isospectral potential families are shown in Fig. 3.

4 Concluding remarks

In the present work, we have proposed a method to find solvable potentials as well as their new generalized

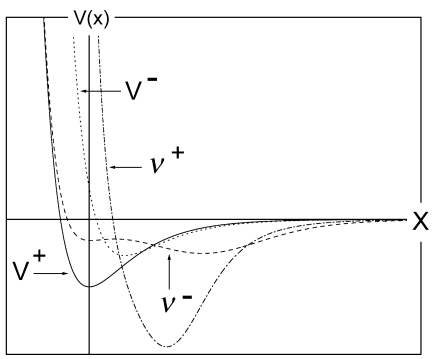

Fig. 3. Morse isospectral potential partners given by Eq. (79) ($\alpha = D = 1$), Eq. (82) ($\alpha = 1$, $D = 3$), Eq. (87) ($\alpha = 1$, $D = 3$, $\gamma = 20b$) and Eq. (88) ($\alpha = D = 1$, $\beta = a/2$) using, as usual, $(0, x)$ as integration limits

isospectral partners. Instead of solving the Riccati equation involved for a specific potential, as is usually done, our proposal is based on the use of the standard Witten superpotential as a particular solution in order to identify the potential under study. Next, we found the general solution of the Riccati equation leading to the corresponding generalized isospectral potential. Thus, with the aim to find new, generalized isospectral potentials, we put the Witten superpotential in terms of reciprocal wavefunctions. With this new representation, the reciprocal Witten superpotential was generalized, leading to the corresponding generalized isospectral potential partners. Also, from the Witten generalized superpotential we obtained the wavefunctions that are solutions of the Schrödinger equation involved. In short our proposal uses the following algorithm: a standard Witten superpotential is used to identify the former solvable potential, the ground-state wavefunction and factorization energy, a generalized Witten superpotential to get the Darboux potential and corresponding wavefunction and a generalized reciprocal Witten superpotential to obtain new generalized isospectral potentials and wavefunctions. The advantage of our proposal stems from the fact that while other approaches are developed with the purpose to solve the Riccati equation for a known potential, in our case the standard and reciprocal Witten superpotentials, which are used as an ansatz to identify the solvable potential under consideration, can be generalized in order to obtain generalized isospectral potentials with their respective generalized wave functions. As an example of the usefulness of the proposed approach, we considered explicitly some standard potentials with the objective to find the corresponding generalized isospectral partners. However, our method is general and can be used straightforwardly for the generalization of other standard solvable potentials as well as to find new solvable potentials that can be used as models in different quantum chemical applications.

Acknowledgement. This work was supported by CONACYT-Mexico, under scientific project No. 32762-E.

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